

A GEOMETRIC VIEW OF THE CHERN CHARACTER

FLORIN DUMITRESCU

ABSTRACT. In this note we show that the Chern character form of a superconnection is obtained via the parallel transport of the superconnection along superpaths, by restriction to the universal superpoint path.

Consider a $\mathbf{Z}/2$ -graded vector bundle E over a manifold M and let \mathbb{A} be a superconnection on E , i.e. an odd first-order differential operator on E that satisfies the graded Leibniz rule:

$$\mathbb{A} : \Omega^*(M, E) \rightarrow \Omega^*(M, E) \quad \mathbb{A}(\omega s) = d\omega \otimes s + (-1)^{\deg \omega} \omega \otimes \mathbb{A}(s),$$

for $\omega \in \Omega^*(M)$ and $s \in \Gamma(M, E)$. This notion generalizes the concept of a connection on a vector bundle in the sense that \mathbb{A} can be split as a connection ∇ on E (grading-preserving) and a linear part $A \in \Omega^*(M, \text{End } E)^{\text{odd}}$. Superconnections were introduced by Quillen in [8] in order to obtain a local representation of the Chern character that would be better suited for a local form of the family index theorem, see [1]. This problem was eventually solved by Bismut [3] making use of the so-called Bismut superconnection in the infinite-dimensional setting (which we do not address here).

This article builds up on an idea of Fei Han [7] to obtain the Chern character form of a connection from the parallel transport along superpaths in the base space M determined by the connection, as defined in [5]. Moreover, Han shows that by considering parallel transport along superloops in M , one obtains the Bismut-Chern character form of [2], an equivariant form on LM the loop space of M which satisfies a differential equation (see Definition 6.4 of [6]) reproduced by parallel transport along superloops. When restricted to M (i.e. constant loops of M) this form gives the ordinary Chern character. Fei Han's work on Bismut-Chern character was explained to me by Stephan Stolz.

Given a superconnection \mathbb{A} on a $\mathbf{Z}/2$ -bundle over a manifold M , we defined in [5] a notion of parallel transport along (families of) superpaths $c : \mathbf{R}^{1|1} \times S \rightarrow M$ (parametrized by arbitrary supermanifolds S) which is compatible with glueing of superpaths, and is invariant under conformal reparametrizations of superpaths in the inverse adiabatic limit. This is done as follows. First, we write $\mathbb{A} = \nabla + A$, with $\nabla = \mathbb{A}_1$ the connection part of the superconnection \mathbb{A} (which shifts the grading by 1 in $\Omega^*(M, E)$) and

$A \in \Omega^*(M, \text{End } E)^{\text{odd}}$ the linear part of the superconnection. For an arbitrary superpath c in M consider the diagram

$$\begin{array}{ccccc}
 E & & & & c^*E \\
 \downarrow & \swarrow & \pi^*E & \nwarrow & \downarrow \\
 M & \xleftarrow{c} & \mathbf{R}^{1|1} \times S & \xrightarrow{\quad} & \mathbf{R}^{1|1} \times S \\
 & \searrow \pi & \downarrow & \swarrow \tilde{c} & \\
 & & \Pi TM & &
 \end{array}$$

with \tilde{c} a canonical lift of the path c to ΠTM , the “odd tangent bundle” of M . Recall that the superpath c can be viewed as a map $c : \mathbf{R} \times S \rightarrow \Pi TM$, via the identification

$$\mathbf{SM}(\mathbf{R}^{1|1} \times S, M) \cong \mathbf{SM}(\mathbf{R} \times S, \Pi TM).$$

Then \tilde{c} is defined as the composition

$$\mathbf{R}^{1|1} \times S = \mathbf{R} \times \mathbf{R}^{0|1} \times S \longrightarrow \mathbf{R}^{0|1} \times \Pi TM \longrightarrow \Pi TM,$$

of the map c followed by the action map T of $\mathbf{R}^{0|1}$ on ΠTM , which, infinitesimally is given by the odd derivation (vector field) d of functions on ΠTM (which are differential forms on M).

Then *parallel transport along c* is defined by *parallel* sections $\psi \in \Gamma(c^*E)$ along c which are solutions to the following differential equation

$$(c^*\nabla)_D\psi - (\tilde{c}^*A)\psi = 0.$$

Here $D = \partial_\theta + \theta\partial_t$ denotes the standard (right invariant) vector field on $\mathbf{R}^{1|1}$, which generates the (super) Lie algebra of the super Lie group $\mathbf{R}^{1|1}$ and whose square is

$$D^2 = \frac{1}{2}[D, D] = \partial_t,$$

the standard time translation vector on \mathbf{R} . See the standard reference [4] or Section 2 of [5] for a brief introduction to supermanifolds.

Our main result here is to show that this “1|1-parallel transport” obtained from a superconnection, when restricted to the “0|1-parallel transport”, it reproduces the Chern character form of the superconnection.

Theorem 1. *Let \mathbb{A} be a superconnection on a $\mathbf{Z}/2$ -bundle E over a manifold M . The 1|1-parallel transport along the superpath given by the composition*

$$\mathbf{R}^{1|1} \times \Pi TM \longrightarrow \mathbf{R}^{0|1} \times \Pi TM \longrightarrow M,$$

where the first map is given by the projection $\mathbf{R}^{1|1} \rightarrow \mathbf{R}^{0|1}$ and the second map is the “superpoint evaluation map” of M , gives rise to the Chern character form of the superconnection $ch(\mathbb{A}) = \text{str}(\exp(-\mathbb{A}^2))$.

Note. In the above, the supertrace str is the extension of the ordinary supertrace defined on $\mathbf{Z}/2$ -graded endomorphisms:

$$\text{str} : \Omega^*(M, E) \longrightarrow \Omega^*(M) : \quad \omega \otimes A \longmapsto \omega \text{ str} A.$$

Proof. Let us begin by remarking that this is a local problem, so it can be reduced to the case of a trivial bundle $E = \underline{\mathbf{R}^p} \oplus \underline{\mathbf{R}^q}$ over M .

We consider first the case when the connection part ∇ of the superconnection \mathbb{A} is *flat*, as the calculation becomes more transparent. In this case, the connection can be taken to be the trivial connection d (there is a trivialization of the bundle E in which the connection is given by the trivial one). Note that the superpath c in M given by the composition $ev \circ p$ lifts to a superpath \tilde{c} in ΠTM given by the composition $T \circ p$ where $T : \mathbf{R}^{0|1} \times \Pi TM \rightarrow \Pi TM$ denotes the left action of $\mathbf{R}^{0|1}$ on ΠTM which on functions is given by

$$\begin{aligned} \Omega^*(M) &\rightarrow \Omega^*(M)[\theta] \\ f &\mapsto f + (df)\theta, \quad \text{for } f \in \Omega^0(M) \\ \alpha &\mapsto \alpha + (-1)^{\deg \alpha} (d\alpha)\theta, \quad \text{for } \alpha \in \Omega^*(M). \end{aligned}$$

The second relation is obtained from the first one, by taking into account that the pushforward of the odd vector field d along the action map T is again d . (The exterior derivative on forms on M is interpreted as an odd vector field or derivation on ΠTM as $\mathcal{C}^\infty(\Pi TM) = \Omega^*(M)$. This vector field squares to zero, giving rise to an $\mathbf{R}^{0|1}$ action- see [5], Section 2.6- on ΠTM , which is the map T .) Let us represent the relevant maps in the diagram

$$\begin{array}{ccccc} & & c & & \\ & \swarrow & & \searrow & \\ M & \xleftarrow{ev} & \mathbf{R}^{0|1} \times \Pi TM & \xleftarrow{p} & \mathbf{R}^{1|1} \times \Pi TM \\ & \nwarrow \pi & \searrow T & \swarrow \tilde{c} & \\ & & \Pi TM & & \end{array}$$

The pullback connection of the trivial connection d along $c = (ev)p$ is still the trivial connection d . The parallel transport equation along c is given by

$$(c^*d)_D(\psi_0 + \theta\psi_1) - (\tilde{c}^*A)(\psi_0 + \theta\psi_1) = 0.$$

As $(c^*d)_D = D$ and $\tilde{c}^*A = A - \theta dA$, the equation becomes

$$(\partial_\theta + \theta\partial_t)(\psi_0 + \theta\psi_1) - (A - \theta dA)(\psi_0 + \theta\psi_1) = 0$$

which is equivalent to

$$\psi_1 + \theta(\partial_t\psi_0) - A\psi_0 + \theta A\psi_1 + \theta(dA)\psi_0 = 0.$$

This gives rise to the system

$$\begin{cases} \psi_1 - A\psi_0 = 0 \\ \frac{\partial\psi_0}{\partial t} + A\psi_1 + (dA)\psi_0 = 0. \end{cases}$$

Combining the last two relations we get

$$\frac{\partial \psi_0}{\partial t} + (dA + A \wedge A)\psi_0 = 0.$$

The solution at $t = 0$ and $t = 1$ defines an element in $\Omega^*(M, \text{End} E)$ given by $\exp(-\mathbb{A}^2)$, as $\mathbb{A}^2 = dA + A \wedge A$. Taking the supetrace of the endomorphism valued form gives the Chern character form of the superconnection.

Let us consider now the general case of an arbitrary superconnection on a (trivial) $\mathbf{Z}/2$ -graded vector bundle. Namely, we have $\mathbb{A} = \nabla + A$ and we write the connection $\nabla = d + \omega$, with $\omega \in \Omega^1(M, \text{End}^0 E)$. Denote by δ the exterior derivative on ΠTM , $\mathbf{R}^{0|1} \times \Pi TM$ and $\mathbf{R}^{1|1} \times \Pi TM$ (as we hope no confusion will arise). The pullback of ∇ via π is

$$\pi^* \nabla = \delta + \pi^* \omega.$$

The pullback of ∇ via $c = (ev)p = \pi T p$ is

$$c^* \nabla = \delta + p^* T^* \pi^* \omega.$$

If $\omega = f dg$, with f, g functions on M , then $\pi^* \omega = f \delta g$. Further,

$$\begin{aligned} T^*(f \delta g) &= T^*(f) \delta T^*(g) \\ &= (f + df\theta) \delta(g + dg\theta) \\ &= (f + df\theta) (\delta g + \delta(dg)\theta - dg\delta\theta) \\ &= f \delta g + f \delta(dg)\theta - f dg\delta\theta - df \delta g \theta + df dg \theta \delta\theta \\ &= f \delta g + (f \delta(dg) - df \delta g) \theta + (-f dg + df dg \theta) \delta\theta. \end{aligned}$$

Therefore,

$$\langle c^*(f dg), \partial_\theta + \theta \partial_t \rangle = -f dg + df dg \theta$$

as g and dg are independent of t and θ . We obtain the formula

$$\langle c^*(\omega), \partial_\theta + \theta \partial_t \rangle = -\omega + (d\omega)\theta.$$

The parallel transport equation along c is given by

$$(c^* \nabla)_D(\psi_0 + \theta \psi_1) - (\tilde{c}^* A)(\psi_0 + \theta \psi_1) = 0.$$

As

$$\langle c^* \nabla, D \rangle = \langle \delta + c^*(\omega), D \rangle = D - \omega + (d\omega)\theta$$

and $\tilde{c}^* A = A - \theta dA$, the equation can be written

$$(\partial_\theta + \theta \partial_t)(\psi_0 + \theta \psi_1) - (\omega - \theta d\omega)(\psi_0 + \theta \psi_1) - (A - \theta dA)(\psi_0 + \theta \psi_1) = 0$$

or

$$(\partial_\theta + \theta \partial_t)(\psi_0 + \theta \psi_1) - (A' - \theta dA')(\psi_0 + \theta \psi_1) = 0$$

where $A' = A + \omega \in \Omega^*(M, \text{End} E)^{odd}$. As before, this is equivalent to

$$\psi_1 + \theta(\partial_t \psi_0) - A' \psi_0 + \theta A' \psi_1 + \theta(dA') \psi_0 = 0.$$

which produces the system

$$\begin{cases} \psi_1 - A'\psi_0 = 0 \\ \frac{\partial\psi_0}{\partial t} + A'\psi_1 + (dA')\psi_0 = 0. \end{cases}$$

Combining these relations we get

$$\frac{\partial\psi_0}{\partial t} + (dA' + A' \wedge A')\psi_0 = 0.$$

The solution at $t = 0$ and $t = 1$ defines an even element in $\Omega^*(M, \text{End } E)$ given by $\exp(-\mathbb{A}^2)$, as $\mathbb{A} = \nabla + A = d + \omega + A = d + A'$ and $\mathbb{A}^2 = dA' + A' \wedge A'$. The supertrace gives us the Chern character form of the superconnection. \square

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Universität Hamburg
 Bundesstraße 55
 D-20146 Hamburg
 Email: florinndo@gmail.com